DISSERTATIO ACADEMICA,

DE MOTU CORPORUM LIBERO IN MEDIO RESISTENTE;

CUJUS PARTEM TERTIAM

VENIA AMPLISS. FACULTATIS PHILOSOPH.

IN IMPERIALI ACADEMIA ABOËNSI,

PUBLICO EXAMINI MODESTE SUBJICIUNT

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Stip. Publ. Satacundensis.

In Audit. phil. die IV Dec. MDCCCXVI.

h. a. m. s.

ABOÆ, Typis FRENCKELLIANIS.

Dato nimirum centro virium, datisque, si res ita postulet, quibusdam quantitatum c, k, l; ea inter D & φ (conjunctim sumtas), curvam descriptam, V, R, v, motum angularem circa centrum virium, motum versus centrum virium, &c. obtinet relatio, ut, datis insuper horum duobus quibuscumque, inveniri semper possint reliqua (exceptis tamen ipsis D & φ). Et sic porro.

Qua quidem ratione problemata hinc oriunda reciproca solvenda sint, ex formulis:

$$R = \varphi\,(D,\, v) = \frac{\sqrt{r^2 - u^2}.\ \mathrm{d}.\ V\dfrac{u^3\mathrm{d}r}{\mathrm{d}u}}{2u^2.r\mathrm{d}r}\,;\quad v = \sqrt{\frac{Vu\mathrm{d}r}{\mathrm{d}u}}\,;$$

$$\text{Motus angul. circa centr. vir.} = \frac{u}{r^2}.\ \sqrt{\frac{Vu\mathrm{d}r}{\mathrm{d}u}}\,;$$

$$\text{Motus versus centr. vir.} = \sqrt{1 - \frac{u^2}{r^2}}.\ \sqrt{\frac{Vu\mathrm{d}r}{\mathrm{d}u}}\,; \quad \text{\&c.}$$

innotescere potest. Observandum vero est, datâ, præter R & v, etiam functione φ, determinari semper posse densitatem medii D, per aequationem $R = \varphi\,(D,\, v)$. Reciproce autem, datis R, v, D, determinata tamen non est forma functionis φ; nisi forsitan, in æquatione: $R = \varphi'\,(D, v, \varphi''\,(v)\,)$, data habeatur functio φ'. Sic ex. gr.,

si

si fuerit $R=D.\,\varphi''(v)$, datæ autem fuerint R, v, D (ut functiones ipsius r), determinari quidem potest φ'', eliminando r inter æquationes $R=D.\,\eta$, $v=\theta$, atque denique quærendo η, in æquatione, quæ hinc oritur, η inter & θ: datâ autem functione φ'', per se patet, determinatam quoque omnino esse formam ipsius φ.

Exemplorum loco, inter quæstiones inversas nuper memoratas, unam alteramve breviter tetigisse non pigebit.

Exemp. 1. Posita utique $R=\alpha Dv^2$, datisque, ipsis α & D, centro virium, quantitatibus c, k, l atque trajectoriâ descriptâ; inveniendæ sint V atque v? Cujus quidem problematis solutionem, per aequationem supra allatam *(17)*, in promtu videbis; unde scil. habentur:

$$V=\frac{c^2k^2.\mathrm{d}u}{u^3\mathrm{d}r.\,e^{-2\alpha\int\frac{Dr\mathrm{d}r}{\sqrt{r^2-u^2}}}},$$

$$v^2\left(=\frac{Vu\mathrm{d}r}{\mathrm{d}u}\right)=\frac{c^2k^2}{u^2\,e^{-2\alpha\int\frac{Dr\mathrm{d}r}{\sqrt{r^2-u^2}}}}.$$

Sit ex. gr. data curva Loganithmica Spiralis, in in cujus centro centrum virium est constitutum; sitque $D = \frac{\beta}{r}$, ubi β constans est quantitas. Erit hoc in casu:

$$u = Ar;$$

unde facili prodeunt negotio:

$$V = \frac{c^2 . r^{\frac{2\alpha\beta}{\sqrt{1-A^2}}-2}}{l^{\frac{2\alpha\beta}{\sqrt{1-A^2}}-2}}, \qquad v = \frac{c. r^{\frac{\alpha\beta}{\sqrt{1-A^2}}-1}}{l^{\frac{\alpha\beta}{\sqrt{1-A^2}}-1}};$$

quibus quidem valoribus, quæsitas V atque v omnino determinatas vides.

Exemp. 2. Ulterius, datis, centro virium ipsisque c, k, l, V atque φ, invenienda sit linea celeritatis æquabilis, mediique in dato quolibet loco densitas? Erit in casu præsente:

$$\frac{Vudr}{du} = c^2, \text{ h. e. } \int Vdr = f(r) = \int \frac{c^2 du}{u} = c^2. \, Log. \; Cu.$$

Hinc:

$$f\,(l) = c^2. \; Log. \; C\,k;$$

 quam

quam utique aequationem, a præcedente, subtrahendo, habetur:

$$f(r) - f(l) = c^2 . \, Log. \, \frac{u}{k}, \text{ h. e. } u = k . e^{\frac{f(r) - f(l)}{c^2}};$$

quæ quæsita igitur est trajectoriæ æquatio.

Hincque erit:

$$R \frac{\left(= \sqrt{r^2 - u^2} \, \mathrm{d} . V \frac{u^3 \mathrm{d}r}{\mathrm{d}u} \right)}{2 \, u^2 . r\mathrm{d}r} = \frac{\sqrt{r^2 - u^2} . V}{r}$$

$$= \left(1 - \frac{k^2}{r^2} . e^{\frac{2f(r) - 2f(l)}{c^2}} \right)^{\frac{1}{2}} . V = \varphi\,(D, c);$$

quæ denique æquatio, resoluta, dat densitatem, quam quæsivimus, D.

Quod si $R = \alpha \, D \, v^2$, $V = \frac{\gamma}{r}$, fiet utique:

$$u = \frac{k . \, r^{\frac{\gamma}{c^2}}}{l^{\frac{\gamma}{c^2}}}, \quad D = \frac{\gamma}{\alpha c \, r} . \sqrt{1 - \frac{k^2}{l^{\frac{2\gamma}{c^2}}} \, r^{\frac{2\gamma}{c^2} - 2}}.$$

Posito

Posito igitur $c^2 = \gamma$, habentur:

$$u = \frac{k}{l}.\, r, \quad D = \frac{\gamma}{ac^2 r}.\sqrt{1 - \frac{k^2}{l^2}},$$

unde videtur, hoc in casu curvam esse Logarithmicam Spiralem, atque densitatem medii formæ $\frac{\beta}{r}$.

Antequam æquatio nostra (16) relinquenda, casum ejus memorabimus præcipua dignum attentione, maximeque ab auctoribus agitatum. Ponatur scilicet densitas medii $D = o$, erit utique (cum neque u, neque dq, $= \infty$ ponendæ sint):

$$d.\, V\frac{u\, dr}{du} = o, \text{ i. e., integrando: } V\frac{u\, dr}{du} = C.$$

Determinata constante $C \left(= V\frac{udr}{du}.\, u^2\right) = c^2 k^2$, fiet utique:

$$Vdr = \frac{c^2 k^2.du}{u^3},$$

atque integrando:

$$\int Vdr = f(r) = C' - \frac{c^2 k^2}{2u^2}.$$

Hinc.

Hinc:

$$f(l) = C' - \frac{c^2}{2};$$

quam quidem æquationem, a priore, subtrahendo, &c., prodibit:

$$u^2 = \frac{[(y-b)\,\mathrm{d}x - (x-a)\,\mathrm{d}y]^2}{\mathrm{d}q^2} = \frac{c^2k^2}{c^2 + 2f(l) - 2f(r)}.$$

Ad separandas denique hac in æquatione variabiles, ponatur:

$$x - a = pr, \quad y - b = r\sqrt{1 - p^2},$$

ubi p nova est variabilis; qua utique substitutione habebitur:

$$\frac{\mathrm{d}p}{\sqrt{1-p^2}} = \frac{\pm\, ck.\,\mathrm{d}r}{r^2.\sqrt{c^2 + 2f(l) - 2f(r) - \frac{c^2k^2}{r^2}}};$$

sive, integrando:

$$Arc.\,Sin.\,p = \int \frac{\pm\, ck.\,\mathrm{d}r}{r^2\sqrt{c^2 + 2f.(l) - 2f(r) - \frac{c^2k^2}{r^2}}} + Const.,$$

h. e

h. e.,

$$Arc. Sin. \left(\frac{x-a}{\sqrt{(x-a)^2+(y-b)^2}} \right)$$

$$= F\left(\sqrt{(x-a)^2+(y-b)^2}\right) + Const.;$$

qua quidem æquatione natura trajectoriæ descriptæ plene est determinata.

Ad velocitatem atque tempus quod attinet, patet, in casu præsente, esse:

$$v^2 \left(= V\frac{udr}{du} \right) = \frac{c^2k^2}{u^2} = c^2 + 2f(l) - 2f(r);$$

atque:

$$dt^2 = \frac{u^2dq^2}{c^2k^2} = \frac{dr^2}{\frac{c^2k^2}{u^2} - \frac{c^2k^2}{r^2}} = \frac{dr^2}{c^2 + 2f(l) - 2f(r) - \frac{c^2k^2}{r^2}}.$$

Quarum utique æquationum, sequentes:

$$v = \frac{ck}{u},\ t = \int \frac{udq}{ck} + Const.,$$

theoremata illa, in doctrinâ virium centripetarum, in vacuo agentium, notissima, continent, quibus quidem *velocitas vera perpendiculis, a centro virium in*

tan-

tangentes trajectoriæ demissis, inverse habetur proportionalis, atque *tempora, quibus diversæ trajectoriæ partes describuntur, arearum circa centrum virium descriptarum directam sequuntur rationem.*

Exemp. Sit $V = \frac{\gamma}{r^2}$; quæritur natura trajectoriæ, in vacuo, descriptæ? Erit hoc in casu: $f(r) = -\frac{\gamma}{r}$; unde:

$$\text{Arc. Sin.}\left(\frac{x-a}{r}\right) = \int \frac{\pm ck.\,dr}{r^2\sqrt{c^2 - \frac{2\gamma}{l} + \frac{2\gamma}{r} - \frac{c^2k^2}{r^2}}},$$

$$= \text{Arc. Sin.} \pm \left\{\frac{\gamma r - c^2k^2}{r.\sqrt{\gamma^2 + c^2k^2\left(c^2 - \frac{2\gamma}{l}\right)}}\right\} + \text{Const.},$$

(adhibitâ, in integrando, substitutione:

$$\frac{1}{r} = s + \frac{\gamma}{c^2k^2}).$$

Ad determinandam constantem arbitrariam, sit: $x = o$, $y = o$; unde:

Arc.

$$\text{Arc. Sin.}\left(\frac{-a}{\sqrt{a^2+b^2}}\right)$$

$$= \text{Arc. Sin.} \pm \left\{\frac{\gamma\sqrt{a^2+b^2}-c^2k^2}{\sqrt{a^2+b^2}\cdot\sqrt{\gamma^2+c^2k^2\left(c^2-\frac{2\gamma}{l}\right)}}\right\} + \text{Const.}$$

Positis igitur:

$$b = o; \quad \pm\left\{\frac{\gamma a - c^2k^2}{a.\sqrt{\gamma^2+c^2k^2\left(c^2-\frac{2\gamma}{l}\right)}}\right\} = -1,$$

$$\text{i. e.} \quad a = \frac{-\gamma \pm \sqrt{\gamma^2+c^2k^2\left(c^2-\frac{2\gamma}{l}\right)}}{c^2-\frac{2\gamma}{l}};$$

quo pacto patet constantem arbitrariam evanescere; substituantur hi denique των a & b valores in æquatione:

$$\text{Arc. Sin.}\left\{\frac{x-a}{\sqrt{(x-a)^2+(y-b)^2}}\right\}$$

$$= \text{Arc. Sin.} \pm \left\{\frac{\gamma\sqrt{(x-a)^2+(y-b)^2}-c^2k^2}{\sqrt{(x-a)^2+(y-b)^2}\cdot\sqrt{\gamma^2+c^2k^2\left(c^2-\frac{2\gamma}{l}\right)}}\right\};$$

 debi-

debitâque adhibitâ reductione, prodibit:

$$y^2 - \frac{c^2 k^2}{\gamma^2}\left(c^2 - \frac{2\gamma}{l}\right). x^2 - \frac{2c^2 k^2}{\gamma}. x = o.$$

Patet igitur, trajectoriam jam descriptam, in omni casu, *Sectionem* esse *Conicam;* & quidem Ellipsin, Parabolam vel Hyperbolam, prout est:

$$c^2 < \frac{2\gamma}{l}, \quad c^2 = \frac{2\gamma}{l}, \quad c^2 > \frac{2\gamma}{l}.$$

Problemata reciproca de viribus centripetis in vacuo agentibus proponenda, quamvis digna memoratu, heic tamen adferre non vacat: haec igitur missa facientes, ad alia transeamus.

§. VIII.

Quæ in præcedentibus ex æquatione nostra (III) hausimus, eam respiciunt hypothesin, qua neque L, neque M, evanescat. Ut eos etiam casus, quibus alterutra tantum potentiarum L & M sollicitet, illustremus, ponatur ex. gr. $L = o$, sitque $M = -P$; examinandum, quid hoc in casu nos

nos doceat æquatio illa (III). Positâ quidem $dx =$ *const.*, faciem induet sequentem satis simplicem:

$$\varphi \left(D, dq\sqrt{-\frac{P}{d^2y}}\right) = -\frac{Pdy}{dq} - \frac{d.\left(\frac{Pdq^2}{-d^2y}\right)}{2dq},$$

$$= \tfrac{1}{2}\, dq.\, d.\left(\frac{P}{d^2y}\right) \;.\;.\;.\;.\;.\; (18);$$

quam quidem æquationem revera tam accedente corpore ad lineam abscissarum, quam discedente, valere, facile perspicuum est.

Valores autem $\tau\omega\nu$ v^2 & dt^2 in hos abeunt:

$$v^2 = -\frac{Pdq^2}{d^2y}, \quad dt^2 = -\frac{d^2y}{P};$$

Exempla quod attinet, directam æquationis (18) tractationem illustrantia, patet utique, in hypothesi $R = \alpha Dv^2$, &, in genere, $R = \alpha Dv^m$, eamdem fere methodum, qua æquationem supra allatam (16) ad inferiorem quasi gradum depressimus, heic quoque adhiberi posse: hac vero ratione, ad curvas quæsitas cognoscendas parum proficientes, casum potius æquationis nostræ sequentem, valde quidem specialem, at usûs (ut facile videbitur) non

non exigui, exempli loco, contemplandum sumemus.

Ponatur quidem $R=\alpha Dv^2$, ubi densitas D constans, fiatque $P=g$, quæ quantitas quoque sit constans; quæritur hoc in casu motus corporis projecti? Erit igitur jam:

$$-\frac{\alpha D dq^2 . g}{d^2 y} = \tfrac{1}{2} dq . d.\left(\frac{g}{d^2 y}\right) = \frac{-g dq . d^3 y}{2 d^2 y^2},$$

h. e.

$$2\alpha D dq . d^2 y = d^3 y \quad . \quad . \quad . \quad . \quad . \quad (19);$$

quæ quidem æquatio differentialis, quamvis una vel altera integratione ad inferiorem reduci gradum possit, optime tamen naturam trajectoriæ quæsitæ illustrabit, per methodum seriei infinitæ, quam proxime, integrata. Positis igitur (quod generatim quoque in sequentibus valeat) x & y simul evanescentibus, accipiatur adeo, ad relationem quæsitam x inter & y definiendam, æquatio sequens, numerum terminorum infinitum habens:

$$y = a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \&c.,$$

designantibus utique coëfficientibus a_1, a_2, &c. quantitates constantes, nondum vero determinatas. Ad quas quidem, quæstioni jam occurrenti conveni-

enter,

enter, investigandas, valorem nuper assumtum ipsius y, ejusque differentialia (in hypothesi præsenti: $dx = const.$, sumta), in æquatione nuper inventa (19), substitui, necesse est; quo igitur calculo subducto, eamdem utique (19) sequenti prodire facie videbis:

$$6a_3 + 24a_4x + 60a_5x^2 + \&c. =$$

$$4\alpha Da_2(1+a_1^2)^{\frac{1}{2}} + \Big(12\alpha Da_3(1+a_1^2)^{\frac{1}{2}} + \frac{8\alpha Da_1a_2^2}{(1+a_1^2)^{\frac{1}{2}}}\Big).x +$$

$$\Big(24\alpha Da_4(1+a_1^2)^{\frac{1}{2}} + \frac{36\alpha Da_1a_2a_3 + 8\alpha Da_2^3}{(1+a_1^2)^{\frac{1}{2}}}$$

$$- \frac{8\alpha Da_1^2a_2^3}{(1+a_1^2)^{\frac{3}{2}}}\Big).x^2 + \&c.;$$

qua quidem æquatione, necessario identicâ, valores ipsarum a_3, a_4, a_5, &c. facile determinari posse, apparet, si modo dentur ipsæ:

$$a_1, \quad a_2,$$

quas utique aliâ inveniendas esse ratione, manifestum est. Fiet autem hoc, si observemus, esse:

$$\frac{dy}{dx} = \frac{dy}{dq} \times \frac{dq}{dx} = a_1 + 2a_2x + 3a_3x^2 + \&c.,$$

$$\frac{d^2y}{2dx^2} = -\frac{gdq^2}{2v^2dx^2} = a_2 + 3a_3x + 6a_4x^2 + \&c.;$$

unde, positis:

$$\frac{dy}{dq} = m, \quad \frac{dx}{dq} = n, \quad v^2 = c^2,$$

in puncto trajectoriæ, quo $x = o$ (ideoque etiam $y = o$), habentur utique:

$$a_1 = \frac{m}{n}, \quad a_2 = -\frac{g}{2c^2n^2}$$

Quos quidem valores, in expressionibus των a_3, a_4, a_5, &c., æquationis allatæ ope inventis, substituendo, cognitas omnino hasce videbis quantitates, quæsitumque ipsius y valorem habebis:

$$y = \frac{m}{n}.x - \frac{g}{2c^2n^2}.x^2 - \frac{\alpha Dg}{3c^2n^3}.x^3$$

$$-\left(\frac{\alpha^2D^2g}{6c^2n^4} - \frac{\alpha Dg.^2m}{12c^4n^4}\right).x^4$$

$$-\left(\frac{\alpha^3D^3g}{15c^2n^5} - \frac{2\alpha^2D^2g^2.m}{15c^4n^5} + \frac{\alpha Dg^3}{60c^6n^5}\right).x^5 - \&c.;$$

quam

quam quidem seriem (quando αD parva fuerit quantitas & c magna, satis convergentem), tam pro descensu corporis, quam ascensu, valere, observandum bene est.

Provenient hinc facile:

$$v^2 = -\frac{g\mathrm{d}q^2}{\mathrm{d}^2y} = c^2 - \left(\frac{2\alpha Dc^2}{n} + \frac{2gm}{n}\right).x + \left(\frac{2\alpha^2 D^2 c^2}{n^2} + \frac{3\alpha Dgm}{n^2} + \frac{g^2}{c^2n^2}\right).x^2 - \left(\frac{4\alpha^3 D^3.c^2}{3n^3} + \frac{8\alpha^2 D^2 g\,m}{3n^3} + \frac{5\alpha Dg^2m^2}{3c^2n^3} + \frac{\alpha Dg^2}{3\,c^2n}\right).x^3 + \&c.;$$

nec non:

$$t = \int\sqrt{\frac{-\mathrm{d}^2y}{g}} = Const. + \frac{x}{cn} + \frac{\alpha D}{2cn^2}.x^2 + \left(\frac{\alpha^2 D^2}{6cn^3} - \frac{\alpha Dgm}{6c^3n^3}\right).x^3 + \left(\frac{\alpha^3 D^3}{24cn^4} - \frac{5\alpha^2 D^2 gm}{24\,c^3n^4} + \frac{\alpha Dg^2}{24c^5n^2}\right).\ x^4 + \&c.;$$

quibus quidem formulis velocitatem quoque projectilis, atque tempus t, pro dato quodam ipsius x valore, computare licet.

Hoc vero jam casu particulari relicto, pro instituti ratione, inversas quasdam quæstiones me-

more-

moremus, quas, in præsenti hypothesi virium absolutarum $L = o$, $M = - P$, proponi posse, notari convenit. Observationem igitur sequentem, allatæ jam supra in doctrina virium centripetarum non absimilem, hoc proponere loco non pigebit: Datâ potentiæ absolutæ directione (datisque, si necesse fuerit, quibusdam nuper memoratarum c, m, n), is quidem, inter D & φ (conjunctim consideratas), trajectoriam descriptam, potentiam absolutam, resistentiam, velocitatem veram, velocitatem in directione ipsius x, velocitatem in directione ipsius y, &c. locum habebit nexus, ut, datis insuper harum duabus quibuscumque, determinari possint ceteræ, exceptis tantum D & φ; sicque porro. Densitatem vero D vel formam functionis φ definire si volueris, quæ supra p. 31, 32 huc pertinentia attulimus notanda tibi sunt, observanti tantum, expressiones ipsarum R, v, D, si vel ambas continerent x & y, per æquationem datam curvæ, ad functiones unius tantum harum coordinatarum semper transformari posse.

Quæ citatis nuper principiis prodeant, problemata reciproca, per æquationes huc spectantes:

$$R = \varphi\,(D,v) = \tfrac{1}{2}\,\mathrm{d}q.\ \mathrm{d}.\left(\frac{P}{\mathrm{d}^2 y}\right);\quad v = \mathrm{d}q.\ \sqrt{-\frac{P}{\mathrm{d}^2 y}};$$

Veloc.